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ON THE LUNAR INEQUALITIES PRODUCED BY THE MOTION OF THE ECLIPTIC.

BY DR. G. W. HILL, Washington, D. C.

[CONTINUED FROM PAGE 10.]

Also we get

$$2\frac{\pi}{n}\frac{ydz - zdy}{d\zeta} = -\left(2 + \frac{1}{12}m^2\right)\gamma\frac{\pi}{n}\cos(\zeta - \eta) - \frac{3}{4}m\gamma\frac{\pi}{n}\cos(\zeta + \eta - 2\tau)$$

$$\begin{aligned} -6m^2(1 + \cos 2\tau)z\delta z = {} & 4\gamma\frac{\pi}{n}\cos(\zeta - \eta) + \left(2 - \frac{3}{4}m\right)\gamma\frac{\pi}{n}\cos(\zeta - \eta - 2\tau) \\ & - 4\gamma\frac{\pi}{n}\cos(\zeta + \eta) + \left(2 - \frac{3}{4}m\right)\gamma\frac{\pi}{n}\cos(\zeta - \eta + 2\tau) \\ & - \left(2 - \frac{9}{4}m\right)\gamma\frac{\pi}{n}\cos(\zeta + \eta - 2\tau) - 2\gamma\frac{\pi}{n}\cos(\zeta + \eta + 2\tau), \end{aligned}$$

$$\begin{aligned} 6m^3\int z\delta z \sin 2\tau\,.\,d\zeta = {} & m\gamma\frac{\pi}{n}\cos(\zeta - \eta - 2\tau) + m\gamma\frac{\pi}{n}\cos(\zeta - \eta + 2\tau) \\ & + \gamma\frac{\pi}{n}\cos(\zeta + \eta - 2\tau), \end{aligned}$$

and, by the addition of these three equations,

$$\begin{aligned} U' = \gamma\frac{\pi}{n}\Big\{ & 2\cos(\zeta - \eta) + \left(2 + \frac{1}{4}m\right)\cos(\zeta - \eta - 2\tau) + \left(2 + \frac{1}{4}m\right)\cos(\zeta - \eta + 2\tau) \\ & - 4\cos(\zeta + \eta) - \left(1 - \frac{3}{2}m\right)\cos(\zeta + \eta - 2\tau) - 2\cos(\zeta + \eta + 2\tau)\Big\}. \end{aligned}$$

In the next place

$$\begin{aligned} 2\frac{\pi}{n}x\frac{dz}{d\zeta} = \gamma\frac{\pi}{n}\Big\{ & \frac{3}{8}m\sin(\zeta - \eta + 2\tau) + \sin(\zeta + \eta) \\ & + \left(\frac{3}{8}m - \frac{21}{32}m^2\right)\sin(\zeta + \eta - 2\tau)\Big\}, \end{aligned}$$

$$3m^2 z\delta z \sin 2\tau = r\frac{\pi}{n}\Big\{ (1 + \tfrac{3}{8}m) \sin(\zeta - \eta - 2\tau) - (1 + \tfrac{3}{8}m) \sin(\zeta - \eta + 2\tau)$$
$$- (1 + \tfrac{3}{8}m) \sin(\zeta + \eta - 2\tau) + \sin(\zeta + \eta + 2\tau) \Big\},$$

$$2\frac{d\lambda}{d\zeta} z\delta z = r\frac{\pi}{n}\Big\{ \left(\frac{1}{2}m^{-1} - \frac{17}{48}m\right) \cos(\zeta - \eta - 2\tau)$$
$$+ \left(\frac{1}{2}m^{-1} - \frac{3}{16} - \frac{55}{384}m\right) \cos(\zeta - \eta + 2\tau)$$
$$+ \left(\frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{29}{9}\right) \cos(\zeta + \eta)$$
$$- \left(m^{-1} + \frac{103}{48} + \frac{8563}{1152}m\right) \cos(\zeta + \eta - 2\tau)$$
$$+ \frac{7}{3} \cos(\zeta + \eta + 2\tau) + \frac{3}{16} \cos(\zeta + \eta - 4\tau) \Big\}.$$

In these expressions the terms depending on the argument $\zeta - \eta$ are omitted because the co-efficient belonging to this argument in $\delta\lambda$ will be determined from the differential equation given specially for this purpose.

Remembering that

$$\frac{d\eta}{d\zeta} = 1 + \frac{3}{4}m^2 - \frac{9}{32}m^3 - \frac{273}{128}m^4,$$

the following expression for U' is readily obtained: —

$$U' = r\frac{\pi}{n}\Big\{ \left(\frac{1}{2}m^{-1} + \frac{1}{2} + \frac{1}{3}m\right) \cos(\zeta - \eta - 2\tau)$$
$$+ \left(\frac{1}{2}m^{-1} + \frac{5}{16} + \frac{137}{384}m\right) \cos(\zeta - \eta + 2\tau)$$
$$+ \left(\frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{49}{18}\right) \cos(\zeta + \eta)$$
$$- \left(\frac{1}{2}m^{-1} + \frac{7}{3} + \frac{839}{144}m\right) \cos(\zeta + \eta - 2\tau)$$
$$+ \frac{25}{12} \cos(\zeta + \eta + 2\tau) + \frac{3}{16} \cos(\zeta + \eta - 4\tau) \Big\}.$$

Let us put now

$$r\delta r = r\frac{\pi}{n}\Big\{ B_1 \cos(\zeta - \eta) + B_2 \cos(\zeta - \eta - 2\tau)$$
$$+ B_3 \cos(\zeta - \eta + 2\tau) + B_4 \cos(\zeta + \eta) + B_5 \cos(\zeta + \eta - 2\tau)$$
$$+ B_6 \cos(\zeta + \eta + 2\tau) \Big\},$$

$$\begin{aligned}\delta\lambda = \gamma\frac{\pi}{n}\Big\{ & C_1 \sin(\zeta - \eta) + C_2 \sin(\zeta - \eta - 2\tau) \\ & + C_3 \sin(\zeta - \eta + 2\tau) + C_4 \sin(\zeta + \eta) + C_5 \sin(\zeta + \eta - 2\tau) \\ & + C_6 \sin(\zeta + \eta + 2\tau) + C_7 \cos(\zeta + \eta - 4\tau)\Big\}.\end{aligned}$$

To a sufficient degree of approximation

$$C = 6m^2 \sin 2\tau,$$

$$D = -\frac{57}{8}m^4 - \frac{97}{4}m^5 + (3m^2 - m^4)\cos 2\tau,$$

$$E = 3m^2 \sin 2\tau.$$

Substituting the expressions for $r\delta r$ and $\delta\lambda$ in the differential equations which serve to determine them, the following equations of condition between the coefficients are obtained : —

$$B_1 = 2,$$

$$-(3 - 8m)B_2 + (3m^2 + \frac{3}{2}m^3)C_1 = 2 + \frac{1}{4}m,$$

$$-(3 - 8m)B_3 - (3m^2 + \frac{3}{2}m^3)C_1 = 2 + \frac{1}{4}m,$$

$$3B_4 = 4,$$

$$-B_5 - \left(\frac{3}{2}m^2 + \frac{9}{16}m^3\right)C_4 = 1 - \frac{3}{2}m,$$

$$15B_6 + 3m^2C_4 = 2,$$

$$(2 - 2m + \frac{1}{12}m^2)C_2 - \left(\frac{3}{4}m^2 + \frac{3}{4}m^3\right)C_1 - 2B_2 = -\frac{1}{2}m^{-1} - \frac{1}{2} - \frac{1}{3}m,$$

$$(2 - 2m - \frac{17}{12}m^2)C_3 - \left(\frac{3}{4}m^2 + \frac{3}{4}m^3\right)C_1 + 2B_3 = \frac{1}{2}m^{-1} + \frac{5}{16} + \frac{137}{384}m,$$

$$(2 + \frac{1}{12}m^2)C_4 + 2B_4 = \frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{49}{18},$$

$$\left\{\begin{array}{l}(2m + \frac{3}{4}m^2)C_5 - \left(\frac{3}{4}m + \frac{55}{32}m^2 + \frac{2417}{384}m^3\right)C_4 \\ \quad -\frac{3}{4}mC_7 + 2B_5 + \frac{3}{4}mB_4\end{array}\right\} = -\frac{1}{2}m^{-1} - \frac{7}{3} - \frac{839}{144}m,$$

$$4C_6 - 2m^2C_4 + 2B_6 = \frac{25}{12},$$

$$2C_7 = -\frac{3}{16}.$$

To obtain an equation for determining C_1 we employ the special differential equation we have given for this purpose. Here we have

$$\frac{d\lambda^2}{d\zeta^2} = 1 + \frac{121}{32}m^4 + \left(\frac{11}{2}m^2 + \frac{85}{6}m^3\right)\cos 2\tau,$$

$$-\left(\frac{dr}{rd\zeta}\right)^2 = -2m^4,$$

$$\frac{21}{2}m^2(1+\cos 2\varphi) = \frac{21}{2}m^2 - \frac{231}{16}m^4 + \frac{21}{2}m^2\cos 2\tau,$$

$$\frac{d\lambda^2}{d\zeta^2} - \left(\frac{dr}{rd\zeta}\right)^2 + \frac{21}{2}m^2(1+\cos 2\varphi) = 1 + \frac{21}{2}m^2 - \frac{405}{32}m^4 + \left(16m^2 + \frac{85}{6}m^3\right)\cos 2\tau.$$

Retaining only the term whose argument is $\zeta - \eta$,

$$\left\{\frac{d\lambda^2}{d\zeta^2} - 1 - \left(\frac{dr}{rd\zeta}\right)^2 + \frac{21}{2}m^2(1+\cos 2\varphi)\right\} z\delta z$$
$$= -\left(7 - \frac{11}{8}m - \frac{1535}{192}m^2\right)\gamma\frac{\pi}{n}\cos(\zeta-\eta).$$

In addition

$$\frac{dr}{rd\zeta} = \left(2m^2 + \frac{13}{3}m^3\right)\sin 2\tau,$$
$$\frac{dr}{rd\zeta}\frac{d(z\delta z)}{d\zeta} = -\left(m + \frac{223}{48}m^2\right)\gamma\frac{\pi}{n}\cos(\zeta-\eta),$$
$$-4\frac{\pi}{n}\frac{ydz - zdy}{d\zeta} = \left(4 + \frac{1}{6}m^2\right)\gamma\frac{\pi}{n}\cos(\zeta-\eta).$$

Let us write the series for z

$$z = \gamma\left\{q_1\sin\eta + q_2\sin(2\tau-\eta) + q_3\sin(2\tau+\eta)\right\},$$

then

$$z\delta z = \frac{1}{2}(A_1q_1 - A_2q_2 + A_3q_3)\gamma\frac{\pi}{n}\cos(\zeta-\eta),$$
$$\frac{dz}{d\zeta} = \gamma\left\{\left(1 + \frac{3}{4}m^2 - \frac{9}{32}m^3 - \frac{273}{128}m^4\right)q_1\cos\eta\right.$$
$$\left. + \left(1 - 2m - \frac{3}{4}m^2\right)q_2\cos(2\tau-\eta) + 3q_3\cos(2\tau+\eta)\right\},$$
$$\frac{d.\delta z}{d\zeta} = \frac{\pi}{n}\left\{A_1\cos\zeta - (1-2m)A_2\cos(\zeta-2\tau) + 3A_3\cos(\zeta+2\tau)\right\},$$
$$z\delta z - \frac{dz}{d\zeta}\frac{d.\delta z}{d\zeta} = -\frac{1}{2}\left\{\left(\frac{3}{4}m^2 - \frac{9}{32}m^3 - \frac{273}{128}m^4\right)A_1q_1\right.$$
$$\left. + \left(4m - \frac{13}{4}m^2\right)A_2q_2 + 8A_3q_3\right\}\gamma\frac{\pi}{n}\cos(\zeta-\eta).$$

Substituting in the last equation the values of A_1, q_1, etc., it becomes

$$z\delta z - \frac{dz}{d\zeta}\frac{d.\delta z}{d\zeta} = \left(\frac{1}{2} - \frac{3}{8}m - \frac{253}{96}m^2\right)\gamma\frac{\pi}{n}\cos(\zeta-\eta).$$

Also we have

$$-3m^2(1+3\cos 2\tau)\,r\delta r = \left[3m^2B_1 + \frac{9}{2}m^2(B_2+B_3)\right]\gamma\frac{\pi}{n}\cos(\zeta-\eta),$$

but from the previous equations of condition, $B_1 = 2$, and $B_2 + B_3 = -\frac{4}{3}$, hence

$$-3m^2(1+3\cos 2\tau)\,r\delta r = 0.$$

In addition

$$\frac{dR}{d\lambda} = -\frac{3}{2}m^2\sin 2\tau,$$

$$-7\frac{dR}{d\lambda}\delta\lambda = -\frac{21}{4}m^2(C_2-C_3)\gamma\frac{\pi}{n}\cos(\zeta-\eta),$$

$$r^2\frac{d\lambda}{d\zeta} = 1 - \frac{1}{3}m^2 + \left(\frac{3}{4}m^2 + \frac{3}{4}m^3\right)\cos 2\tau,$$

$$-3m\int\delta.\frac{dR}{d\lambda}d\zeta = 3m\int[D\delta\lambda + E(r\delta r - z\delta z)]\,d\zeta,$$

$$3m\int D\delta\lambda d\zeta = -\left\{\left(\frac{57}{2}m^3 + \frac{1723}{16}m^4\right)C_1 - (6m+\frac{9}{4}m^2)(C_2+C_3)\right\}\gamma\frac{\pi}{n}\cos(\zeta-\eta),$$

$$3m\int Er\delta rd\zeta = (6m+\frac{9}{4}m^2)(B_2-B_3)\gamma\frac{\pi}{n}\cos(\zeta-\eta),$$

$$-3m\int Ez\delta zd\zeta = -\left(\frac{9}{16}m - \frac{27}{64}m^2\right)\gamma\frac{\pi}{n}\cos(\zeta-\eta).$$

Thus is obtained the equation which determines C_1: —

$$\left\{\begin{array}{l}\left(\frac{3}{4}m^2 - \frac{9}{32}m^3 - \frac{305}{128}m^4\right)C_1 \\ -\frac{9}{2}m^2(C_2-C_3) - \left(\frac{57}{2}m^3 + \frac{1723}{16}m^4\right)C_1 \\ +(6m+\frac{9}{4}m^2)(B_2-B_3+C_2+C_3)\end{array}\right\} = \frac{5}{2} + \frac{9}{16}m - \frac{125}{96}m^2.$$

But the previous equations of condition furnish

$$C_2 - C_3 = -\frac{1}{2}m^{-1} - \frac{215}{96},$$

$$B_2 - B_3 + C_2 + C_3 = \left(\frac{19}{4}m^2 + \frac{97}{6}m^3\right)C_1 - \frac{3}{32} + \frac{27}{256}m,$$

consequently

$$\left(\frac{3}{4}m^2 - \frac{9}{32}m^3 - \frac{305}{128}m^4\right)C_1 = \frac{5}{2} - \frac{9}{8}m - \frac{1133}{96}m^2,$$

and

$$C_1 = \frac{10}{3} m^{-2} - \frac{1}{4} m^{-1} - \frac{503}{96}.$$

Solving the remaining equations of conditions we get

$$\begin{aligned}
\delta\lambda = & \left(\frac{10}{3} m^{-2} - \frac{1}{4} m^{-1} - \frac{503}{96}\right) \gamma \frac{\pi}{n} \sin(\zeta - \eta) \\
& - \left(\frac{1}{4} m^{-1} - \frac{41}{12}\right) \gamma \frac{\pi}{n} \sin(\zeta - \eta - 2\tau) \\
& + \left(\frac{1}{4} m^{-1} + \frac{181}{32}\right) \gamma \frac{\pi}{n} \sin(\zeta - \eta + 2\tau) \\
& + \left(\frac{2}{3} m^{-2} + \frac{1}{4} m^{-1} + 0\right) \gamma \frac{\pi}{n} \sin(\zeta + \eta) \\
& + \left(\frac{3}{2} m^{-1} - \frac{235}{96}\right) \gamma \frac{\pi}{n} \sin(\zeta + \eta - 2\tau) \\
& + \frac{41}{48} \gamma \frac{\pi}{n} \sin(\zeta + \eta + 2\tau) \\
& - \frac{3}{32} \gamma \frac{\pi}{n} \sin(\zeta + \eta - 4\tau).
\end{aligned}$$

The expression for the inequalities in latitude is

$$\begin{aligned}
\delta\beta = \frac{\delta z}{r} = & - \left(\frac{4}{3} m^{-2} + \frac{1}{2} m^{-1} + \frac{11}{3} + \frac{3347}{288} m\right) \frac{\pi}{n} \sin \zeta \\
& + \left(\frac{1}{2} m^{-1} + \frac{17}{12} + \frac{1187}{288} m\right) \frac{\pi}{n} \sin(\zeta - 2\tau) \\
& - \left(\frac{11}{12} + \frac{1043}{288} m\right) \frac{\pi}{n} \sin(\zeta + 2\tau) \\
& + \frac{11}{32} m \frac{\pi}{n} \sin(\zeta - 4\tau) \\
& + \left(\frac{4}{3} m^{-2} + \frac{1}{2} m^{-1} + \frac{35}{48}\right) c \frac{\pi}{n} \sin(\zeta - \xi) \\
& - \left(\frac{4}{3} m^{-2} + \frac{1}{2} m^{-1} + \frac{13}{3}\right) c \frac{\pi}{n} \sin(\zeta + \xi) \\
& + \left(2 m^{-1} + 2\right) c \frac{\pi}{n} \sin(\zeta - 2\tau + \xi) \\
& - \left(\frac{5}{2} m^{-1} + \frac{527}{48}\right) c \frac{\pi}{n} \sin(\zeta + 2\tau - \xi) \\
& + \left(\frac{1}{2} m^{-1} + \frac{7}{24}\right) c \frac{\pi}{n} \sin(\zeta - 2\tau - \xi)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{7}{3}e\frac{\pi}{n}\sin(\zeta+2\tau+\xi)\\
&+\frac{15}{16}e\frac{\pi}{n}\sin(\zeta-4\tau+\xi)\,.
\end{aligned}
$$

[TO BE CONTINUED.]

NOTE ON A UNIQUE PROPERTY OF AN AXISYMMETRIC DETERMINANT OF THE FOURTH ORDER.

By PROF. THOMAS MUIR, Bishopton, Scotland.

The determinant in question is

$$\begin{vmatrix} 0 & a & b & c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & k \end{vmatrix}, \text{ or } \Omega \text{ say,}$$

its only specialty as an axisymmetric determinant of the fourth order being that it has a zero in the place (1,1). Now if we take the minor

$$\begin{vmatrix} 0 & a & b \\ a & d & e \\ b & e & g \end{vmatrix}, \text{ or } K \text{ say,}$$

and substitute in it for a and b the complementary minors of h and f in Ω, it is found that, curiously enough, the new determinant is divisible by the old, the quotient being

$$\begin{vmatrix} 0 & a & b & -c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & 0 \end{vmatrix}.$$

That is to say, if H and F denote the complementary minors of h and f in Ω we assert the identity

$$\begin{vmatrix} 0 & H & F \\ H & d & e \\ F & e & g \end{vmatrix} = \begin{vmatrix} 0 & a & b \\ a & d & e \\ b & e & g \end{vmatrix} \cdot \begin{vmatrix} 0 & a & b & -c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & 0 \end{vmatrix}.$$